A Mathematical Model of Icosahedral Materials

by

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Introduction

An icosahedral material is a regular three-dimensional array of icosahedra held together by a repeating pattern of flexible interconnecting bands. Each interconnecting band attaches to exactly two icosahedra and each icosahedron has up to 12 bands attached to it, depending on its position within the material. For interior icosahedra with 12 attached bands, this restrains motions in both directions for all three Cartesian coordinate axes. The approach to understanding the motions of an icosahedral material that I will follow in this document is to assume that the icosahedra are rigid bodies and apply Newtonian physics to describe their motions. I will begin by considering the motions of an icosahedron in the material in response to changes in the positions of its attached interconnecting elements and any forces applied to it. Once I have determined the equations that describe these motions, it will then be possible to describe the motions of the entire material by considering interconnected arrays of icosahedra whose individual motions alter the positions of the ends of the interconnecting elements that are attached to their neighbors. The resulting mathematical model will allow investigation of the behaviors of a material in static or dynamic contexts. Solving the equations of the model, which means constructing approximate solutions on a computer, will be easier in the static case but may require significant computational power and skill in both cases.

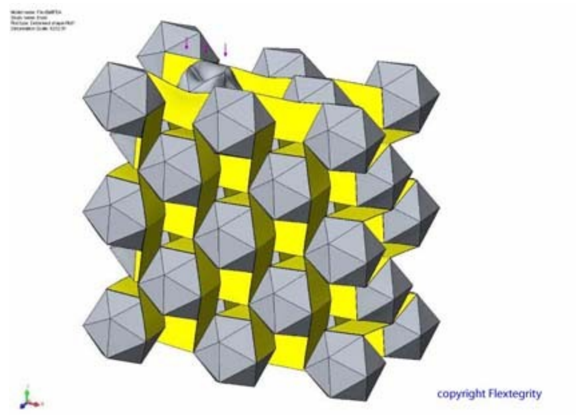

Figure 1: An icosahedral fabric

For the purposes of constructing a tractable mathematical model, I will assume that an icosahedron is a rigid shell of material between two concentric icosahedral surfaces with constant density ρ and uniform thickness T, as measured, for example, at the center of a face. So I am ignoring any additional structure in the interiors of the icosahedra, either assuming that any such structures have too little mass to be important or that the mass of any interior bracing is also icosahedrally distributed.

In general, an icosahedron within an icosahedral material may experience the following forces

- the force of gravity,

- other body forces such as electromagnetic interactions due to a charge distribution within the shell,
- forces on any exterior surfaces of the form $\boldsymbol{F}(\xi,\eta,t)$, where ξ and η are coordinates of points on the surface and t is the time,
- forces due to the motions of the interconnecting elements attached to the icosahedron.

Surface forces include externally applied forces and possibly forces generated when icosahedra collide. If the faces of the icosahedra are frictionless, then only the components of surface forces on the exteriors of icosahedra that are normal to the faces will influence the icosahedra. If, however, the faces exhibit friction, then tangential components will also have effects. The last two forces will usually change over time. The force of gravity will be assumed to remain constant and may or may not be important in a given situation.

Interconnecting elements could be represented as hinges, linear springs with or without a failure point, or elastic bands that attach to points on the faces, edges, or vertices of the icosahedra. In hopes of realistically representing the interconnecting elements in the simplest way possible, I will assume that the interconnecting elements are linear springs that exert forces along the lines connecting the points where their ends attach to icosahedra and that the masses of the elements are small enough to be neglected. Suppose that I_1 and I_2 are icosahedra that are attached by an interconnecting element at points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, where $\boldsymbol{r}_1$ lies on I_1 and $\boldsymbol{r}_2$ lies on I_2. Let D be the natural length of the element and k be its spring constant. Then, in accordance with Newton's third law and Hooke's law, the force that the element exerts will be

$$\boldsymbol{F}_{12}=k\left(\left|\boldsymbol{r}_1-\boldsymbol{r}_2\right|-D\right)\begin{cases}\dfrac{(\boldsymbol{r}_2-\boldsymbol{r}_1)}{\left|\boldsymbol{r}_1-\boldsymbol{r}_2\right|} & \text{on } I_1\\[2ex] \dfrac{(\boldsymbol{r}_1-\boldsymbol{r}_2)}{\left|\boldsymbol{r}_1-\boldsymbol{r}_2\right|} & \text{on } I_2\end{cases} \qquad (1)$$

(This isn't quite right. The force that an interconnecting element exerts also acts to restore its angles of attachment and a particular orientation of the icosahedra it connects relative to each other. It doesn't just act to restore the length of the element. This effect depends on whether a portion of either icosahedron lies between the connection points.) It might also be useful to allow the interconnecting elements to have different stiffnesses depending on whether they are stretched beyond their natural length or compressed shorter than their natural length. In this case,

$$k = \begin{cases} k_e \text{ if } |\boldsymbol{r}_1 - \boldsymbol{r}_2| > D \\ k_c \text{ if } |\boldsymbol{r}_1 - \boldsymbol{r}_2| < D \end{cases}. \quad (2)$$

Furthermore, it's possible to specify that the interconnecting element breaks if stretched too far. In this situation,

$$\boldsymbol{F}_{12} = 0 \text{ for all } t > t_b \text{ if } |\boldsymbol{r}_1 - \boldsymbol{r}_2| = L_e \text{ at } t = t_b \text{ while } |\boldsymbol{r}_1 - \boldsymbol{r}_2| < L_e \text{ for all } t < t_b, \quad (3)$$

where $L_e > D$ is the length at which the interconnecting element breaks under extension and t_b is the time of breakage. This description neglects any change in the properties of the element that might occur near the breaking point and neglects any recoil that might occur afterwards. An analogous condition may be imposed if the element breaks when compressed too far. Notice that, under the above description and consistent with Newton's third law, the force that a connection exerts on the second icosahedron is simply the opposite of the force that it exerts on the first icosahedron.

Theory of Elasticity References: Feynman, Leighton, and Sands V. II, pp. 38-1 to 39-13 and pp. 31-6 to 31-12; Borisenko and Tarapov, pp. 66-8 and 70-2; Strang pp. 155-81; Fetter and Walecka pp. 459-79; and Arfken pp. 140-50.

Equations of Motion for a Rigid Body

The motions of a rigid body are completely specified by Newton's second law

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = m\boldsymbol{a}, \quad (4)$$

which states that the force $\boldsymbol{F}$ is the rate of change of the (linear) momentum $\boldsymbol{p}$, and its rotational counterpart

$$\boldsymbol{N} = \frac{d\boldsymbol{L}}{dt}, \quad (5)$$

which states that the torque $\boldsymbol{N}$ is the rate of change of the angular momentum $\boldsymbol{L}$. Here m is the mass of the body and $\boldsymbol{a}$ is its acceleration. Any quantities written in a bold font or, alternatively, with arrows over them are vector quantities.

In analyzing the motions of a body, it's worthwhile to consider three different coordinate systems, as shown in Figure 1. In this discussion, I will adopt the notation of Fetter and Walecka. (For information related to the following analysis, see Fetter and Walecka pp. 134-9 and pp. 31-9; Fowles pp.117-22 and Chapters 7 and 8; and Bourg Chapters 1, 14,

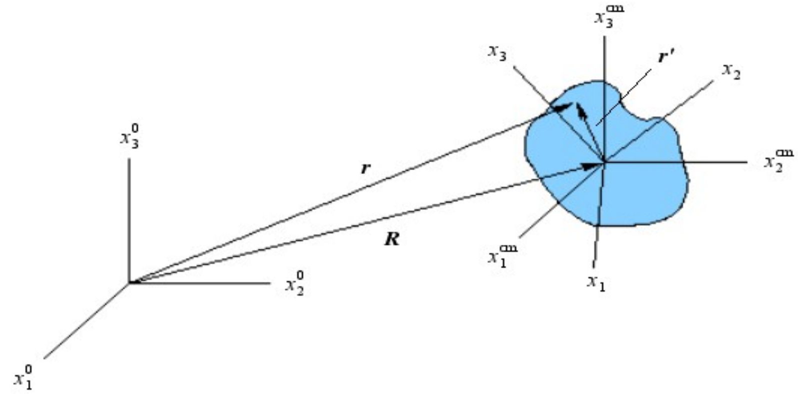

Figure 2: Three coordinate frames

and 15.) I will first introduce a coordinate system associated with the perspective of an inertial observer, typically located outside of the body, who is watching the body move. This coordinate system is a right-handed Cartesian coordinate system with coordinates

$$\{x_i^0\},$$

and corresponding unit basis vectors

$$\{\hat{e}_i^0\},$$

where $i=1,2,3$. I will call this frame the "primary inertial reference frame" or simply "primary frame." This is the coordinate system from which we'd like to determine the body's motions. I define the vector ***R*** to be the position of the center of mass of the body in the primary frame. The center of mass of an icosahedron lies at the center of the icosahedron. So, in this case, ***R*** points from the origin of the primary frame to the center of the icosahedron.

The other two coordinate systems are associated with the body itself and are used to simplify the analysis. The first of these is a coordinate system that has the same unit basis vectors as the primary frame with its origin located at the center of mass of the body. I will call this frame the center-of-mass frame. Its coordinates will be

$$\{x_i^{cm}\},$$

where

$$x_i^{cm}=x_i^0+R_i \text{ for } i=1,2,3. \quad (6)$$

The third coordinate system is a right-handed Cartesian coordinate system with its origin at the center of mass of the body that is attached to the body and fixed so that it moves as the body moves. I will call this frame the body frame and define

$$\{x_i\}$$

to be its coordinates and

$$\{\hat{e}_i\}$$

to be the corresponding unit basis vectors, where again $i=1,2,3$. The position $\boldsymbol{r}$ of an arbitrary point in the body as seen from the primary frame may be written as

$$\boldsymbol{r} = \boldsymbol{R}+\boldsymbol{r}',$$

where $\boldsymbol{r}'$ is the position of the point relative to the center of mass at $\boldsymbol{R}$. For each point in the body, the position $\boldsymbol{r}'$ relative to the center of mass is fixed in the body frame and only depends on the particular point chosen.

The center of mass moves like a point mass, having the mass M of the entire body, that experiences the net force $\boldsymbol{F}_{total}$ applied to the body. Therefore, the position of the center of mass $\boldsymbol{R}$ satisfies Newton's second law

$$M\ddot{\boldsymbol{R}} = \boldsymbol{F}_{total} = \sum_{body} \boldsymbol{F}(\boldsymbol{r},t), \quad (7)$$

where the applied force $\boldsymbol{F}$ may depend on the positions of points on the body and on the time t. In general, the sum will contain volume, area, and line integrals of distributed forces as well as any point forces acting on the body.

The angular momentum $\boldsymbol{L}$ of the body is given by the equation

$$\boldsymbol{L} = \boldsymbol{R}\times(M\dot{\boldsymbol{R}})+\boldsymbol{L}', \quad (8)$$

where $\boldsymbol{L}'$ is the angular momentum of the body about the center of mass. The rotational analog of Newton's second law allows us to calculate how $\boldsymbol{L}'$ changes in response to the applied forces. It implies that

$$\left(\frac{d\boldsymbol{L}'}{dt}\right)_{primary} = \left(\frac{d\boldsymbol{L}'}{dt}\right)_{cm} = \boldsymbol{N}_{total} = \sum_{body} \boldsymbol{r}'\times\boldsymbol{F}(\boldsymbol{r},t), \quad (9)$$

where N_{total} is the net torque. The angular momentum L' may be expressed in terms of the moment of inertia tensor $\boldsymbol{I}$ and the angular velocity $\vec{\omega}$ as

$$\boldsymbol{L}'=I\vec{\omega} \quad (10)$$

so that Equation (9) may be rewritten as

$$\frac{d}{dt}I\vec{\omega} + (\vec{\omega}\times I\vec{\omega}) = \sum_{body} \boldsymbol{r}'\times\boldsymbol{F}(\boldsymbol{r},t). \quad (11)$$

This equation determines the angular velocity of the body $\vec{\omega}$ as a function of time. The components of the moment of inertia tensor are constant in any particular body frame and are given by

$$I_{ij} = \int_V \rho(\boldsymbol{P})(\delta_{ij}\boldsymbol{P}\cdot\boldsymbol{P}-x_i x_j)\, d\, V, \quad (12)$$

for $i,j=1,2,3$, where ρ is the density of the body, δ_{ij} is the Kronecker delta, and the integration ranges over all points $\boldsymbol{P}$ in the body, where $\{x_i\}=\{\boldsymbol{P}\cdot\hat{e}_i\}$ are the components of

$\boldsymbol{P}$ in the chosen body frame. (See Feynmann, Leighton, and Sands V. II, pp. 31-6 to 31-8; Fetter and Walecka pp. 134-7; and Fowles pp. 217-9.) Notice that the moment of inertia tensor is symmetric, which means that its determination only requires the evaluation of 6 of these integrals rather than all 9.

Given specified primary and body frames, Equations (7) and (9) provide the equations of motion for the body

$$M\ddot{\boldsymbol{R}} = \sum_{body} \boldsymbol{F},$$
$$\frac{d}{dt} I\vec{\omega} + (\vec{\omega} \times I\vec{\omega}) = \sum_{body} \boldsymbol{r}' \times \boldsymbol{F}, \quad (13)$$

where $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}'$. These equations are usually coupled since the force usually depends on the position of the body. Aside from these equations, it is also necessary to specify the initial conditions

$$\boldsymbol{R}(0) = \boldsymbol{R}_0,$$
$$\dot{\boldsymbol{R}}(0) = \dot{\boldsymbol{R}}_0, \quad (14)$$
$$\vec{\omega}(0) = \vec{\omega}_0,$$

where $\boldsymbol{R}_0$, $\dot{\boldsymbol{R}}_0$, and $\vec{\omega}_0$ are constant vectors, as well as the forces

$$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, t) \quad (15)$$

that the body will experience. In addition, the motions of the body may be constrained in some way. Any such constraints must be consistent with the initial conditions. System (13) with the initial conditions given in System (14), possibly some constraints, and specified forces provide 6 scalar differential equations that determine $\boldsymbol{R}$ and $\vec{\omega}$ and completely determine the motions of the body.

When working with these equations, it is important to be careful to express vector components in the appropriate coordinate systems. In any single equation, the components of all terms must be expressed in the same coordinates when solving the equation. The equation for the position $\boldsymbol{R}$ of the center of mass is best solved in the primary frame and the equation for the angular velocity $\vec{\omega}$ is best solved in the body frame. Therefore, it is important to be able to express the angular velocity in the center-of-mass frame and the torques in the body frame. Translating back and forth between the primary and center-of-mass coordinate systems is a straightforward matter of adding or subtracting $\boldsymbol{R}$ to or from the position. But the transformation between the body and center-of-mass coordinates is complicated and time-dependent, and it is necessary to solve an additional system of differential equations to determine it.

To find this coordinate transformation, consider the unit basis vectors for the body frame $\{\hat{e}_i\}$. These vectors may be expressed in the center-of-mass frame as

$$\hat{e}_i(t) = \sum_{j=1}^{3} (\hat{e}_i \cdot \hat{e}_j^0)\hat{e}_j^0 = \sum_{j=1}^{3} \mu_{ij}\hat{e}_j^0, \quad (16)$$

where $i=1,2,3$ and

$$\mu_{ij} \equiv \hat{e}_i \cdot \hat{e}_j^0 \text{ for } i, j=1,2,3. \quad (17)$$

If $\{\omega_i\}$ are the components of the angular velocity $\vec{\omega}$ in the body frame, then the angular velocity in the center-of-mass frame is

$$\vec{\omega}(t) = \sum_{i=1}^{3} \omega_i \hat{e}_i = \sum_{i=1}^{3} \omega_i \sum_{j=1}^{3} \mu_{ij}\hat{e}_j^0. \quad (18)$$

Since the unit basis vectors for the body frame do not change in the body frame,

$$\left(\frac{d\hat{e}_i}{dt}\right)_{cm} = \vec{\omega} \times \hat{e}_i, \quad (19)$$

for all i. This equation provides 9 differential equations that determine the coefficients μ_{ij} for $i, j=1,2,3$ and describe the rotation of the body frame in the center of mass frame given the initial positions of the body frame unit vectors in the center-of-mass frame

$$\hat{e}_i(0) = \sum_{j=1}^{3} \mu_{ij}(0)\hat{e}_j^0 \text{ for } i=1,2,3. \quad (20)$$

Since these equations contain the angular velocity, they are coupled with the equations of motion and must be solved simultaneously with the previous equations. This approach requires solving 15 scalar differential equations to determine the motions of a rigid body and, in particular, of each icosahedron.

Rather than using the coefficients μ_{ij} to specify the rotation of the body coordinate system, it is possible to describe rotations using mathematical entities called quaternions, which can be thought of as a generalization of complex numbers. (See Kuipers, Bourg pp. 227-9, Behnke et al pp. 467-77, Penrose pp. 198-208, and the Wikipedia and Wolfram MathWorld entries pertaining to quaternions.) A rotation of angle θ about an axis that points in the direction of the unit vector $\boldsymbol{u}$ is represented by the unit quaternion

$$\boldsymbol{q} = \left[\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\boldsymbol{u}\right] = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(u_1\hat{i} + u_2\hat{j} + u_3\hat{k}). \quad (21)$$

In the center of mass coordinate frame, the quaternion that represents the rotation of the

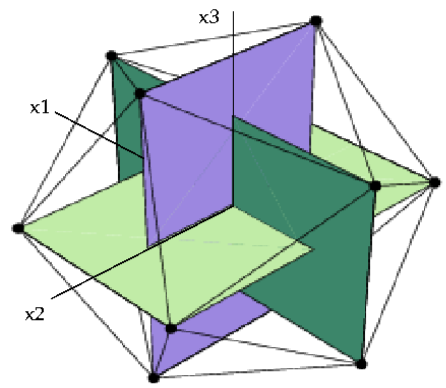

Figure 3: An icosahedron and its body frame

body frame satisfies

$$\frac{d\boldsymbol{q}}{dt} = \frac{1}{2}\boldsymbol{\omega}\boldsymbol{q}\,, \quad (22)$$

where the angular velocity is written as the "pure" quaternion

$$\boldsymbol{\omega} = [0, \vec{\omega}]. \quad (23)$$

In the body frame, $\boldsymbol{q}$ satisfies the equation

$$\frac{d\boldsymbol{q}}{dt} = \frac{1}{2}\boldsymbol{q}\,\boldsymbol{\omega}\,, \quad (24)$$

where $\vec{\omega}$ is written in the body frame instead. The equations are different since quaternions do not commute under multiplication. The solution to either equation must satisfy the initial condition

$$\boldsymbol{q}(0) = \boldsymbol{q}_0\,, \quad (25)$$

where $\boldsymbol{q}_0$ is a constant quaternion expressed in the appropriate coordinate system. If a point with initial position $\boldsymbol{r}'_0$ in the body has been rotated according to a quaternion $\boldsymbol{q}$, then its new position is

$$\boldsymbol{r}' = \boldsymbol{q}\,\boldsymbol{r}'_0\boldsymbol{q}^*\,, \quad (26)$$

where $\boldsymbol{q}^*$ is the conjugate

$$\boldsymbol{q}^* = q_0 - q_1\hat{i} - q_2\hat{j} - q_3\hat{k} \quad (27)$$

of $\boldsymbol{q}$. Using quaternions reduces the number of equations required for determining the coordinate transformation from 9 to 4, which reduces the number of equations required for determining the motions of a rigid body and of an icosahedron, in particular, from 15 to 10.

Mathematical Representation of Icosahedral Shells

An icosahedron is conveniently represented in a right-handed Cartesian coordinate system that has its origin at the center of the icosahedron and has axes that pass through the centers of three perpendicular pairs of edges. (See the Wikipedia and Wolfram MathWorld references related to the icosahedron.) For icosahedra, I define this to be the body coordinate system that I described above. Another way of looking at this is that the axes of the body coordinate system are the mid-lines of three perpendicular golden rectangles that define the icosahedron, as depicted in Figure 3.

An icosahedron with edges of length 2a has vertices of the form

$$(0, \pm a, \pm\tau a), (\pm a, \pm\tau a, 0), (\pm\tau a, 0, \pm a)$$

in this coordinate system, where τ is the golden ratio = $(1+\sqrt{5})/2 \approx 1.618033989$. (See the Wikipedia and Wolfram MathWorld entries regarding the icosahedron and the Wikipedia entries regarding the golden ratio and golden rectangles.) The vertices may be categorized according to their z-components, as follows.

1. Top vertices: $(0,-a,\tau a),(0,a,\tau a)$
2. Second highest vertices: $(-\tau a,0,a),(\tau a,0,a)$
3. Third highest vertices: $(-a,-\tau a,0),(-a,\tau a,0),(a,-\tau a,0),(a,\tau a,0)$
4. Fourth highest vertices: $(-\tau a,0,-a),(\tau a,0,-a)$
5. Bottom vertices: $(0,-a,-\tau a),(0,a,-\tau a)$

The faces may be identified by their vertices and may also be categorized according to their z-components. Furthermore, it is useful to number the faces. Therefore, I may somewhat arbitrarily number the faces as follows

- Top faces:
  1. $\{(0,-a,\tau a),(0,a,\tau a),(-\tau a,0,a)\}$,
  2. $\{(0,-a,\tau a),(0,a,\tau a),(\tau a,0,a)\}$,
- Second highest faces:
  3. $\{(0,-a,\tau a),(-\tau a,0,a),(-a,-\tau a,0)\}$,
  4. $\{(0,-a,\tau a),(-a,-\tau a,0),(a,-\tau a,0)\}$,
  5. $\{(0,-a,\tau a),(a,-\tau a,0),(\tau a,0,a)\}$,
  6. $\{(0,a,\tau a),(-\tau a,0,a),(-a,\tau a,0)\}$,
  7. $\{(0,a,\tau a),(-a,\tau a,0),(a,\tau a,0)\}$,

8. $\{(0, a, \tau a), (a, \tau a, 0), (\tau a, 0, a)\}$,

- Third highest faces:

9. $\{(-\tau a, 0, a), (-a, -\tau a, 0), (-\tau a, 0, -a)\}$,
10. $\{(-\tau a, 0, a), (-a, \tau a, 0), (-\tau a, 0, -a)\}$,
11. $\{(\tau a, 0, a), (a, -\tau a, 0), (\tau a, 0, -a)\}$,
12. $\{(\tau a, 0, a), (a, \tau a, 0), (\tau a, 0, -a)\}$,

- Fourth highest faces:

13. $\{(0, -a, -\tau a), (-\tau a, 0, -a), (-a, -\tau a, 0)\}$,
14. $\{(0, -a, -\tau a), (-a, -\tau a, 0), (a, -\tau a, 0)\}$,
15. $\{(0, -a, -\tau a), (a, -\tau a, 0), (\tau a, 0, -a)\}$,
16. $\{(0, a, -\tau a), (-\tau a, 0, -a), (-a, \tau a, 0)\}$,
17. $\{(0, a, -\tau a), (-a, \tau a, 0), (a, \tau a, 0)\}$,
18. $\{(0, a, -\tau a), (a, \tau a, 0), (\tau a, 0, -a)\}$,

- Bottom faces:

19. $\{(0, -a, -\tau a), (0, a, -\tau a), (-\tau a, 0, -a)\}$,
20. $\{(0, a, -\tau a), (0, -a, -\tau a), (\tau a, 0, -a)\}$.

A face with vertices $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, and $\boldsymbol{v_3}$ is the set of points

$$\{\boldsymbol{v}_1 + \lambda_2(\boldsymbol{v}_2 - \boldsymbol{v}_1) + \lambda_3(\boldsymbol{v}_3 - \boldsymbol{v}_1) \mid 0 \leqslant \lambda_2 + \lambda_3 \leqslant 1\}. \quad (28)$$

Here, the parameters λ_2 and λ_3 are coordinates on the plane spanned by the vectors

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \boldsymbol{v}_1, \ \ \boldsymbol{w}_3 = \boldsymbol{v}_3 - \boldsymbol{v}_1 \quad (29)$$

with origin at

$$\boldsymbol{w}_1 = \boldsymbol{v}_1. \quad (30)$$

The coordinate λ_2 is the coordinate in the direction of the vector $\boldsymbol{w_2}$ and the coordinate λ_3 is the coordinate in the direction of the vector $\boldsymbol{w_3}$. A third coordinate λ_1 specifies distance in the $\boldsymbol{w_1}$ direction. The coordinates of the body frame x_1, x_2, and x_3 are related to the coordinates λ_1, λ_2, and λ_3 via the linear transformation

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = [\boldsymbol{w}_1\, \boldsymbol{w}_2\, \boldsymbol{w}_3] \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = A\vec{\lambda}, \quad (31)$$

where the columns of the transformation matrix A are the vectors $\boldsymbol{w_1}$, $\boldsymbol{w_2}$, and $\boldsymbol{w_3}$. As defined, the vectors $\boldsymbol{w_1}$, $\boldsymbol{w_2}$, and $\boldsymbol{w_3}$ are linearly independent, which means that the inverse transformation

$$\vec{\lambda} = A^{-1}\boldsymbol{x} \quad (32)$$

exists.

For the purposes of the model and for calculating the physical properties of an icosahedral shell, it is important to understand how to integrate over the volume of the shell. Integrating over the entire shell is most easily accomplished by integrating over each part of the shell that corresponds to a face and adding the results. Following this procedure, integrating over the shell entails performing 20 integrals, each one pertaining to one of the 20 faces.

The portion of an icosahedral shell that corresponds to a face is a frustum of the triangular pyramid formed by the vertices of the outer face and the center of the shell, as shown in Figure 3. Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ be the vertices of the outer face and suppose that the length of the edges of the outer face is $2a$ and the length of the edges of the inner face is $2b$. Since the vertices of the inner face lie in the directions of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ from the center of the shell, it is possible to use the coordinates λ_1, λ_2, and λ_3 that pertain to the outer surface to integrate over the frustum. At the outer face, $\lambda_1 = 1$ and, at the inner face, $\lambda_1 = b/a$. On any intermediate surface corresponding to a fixed value of λ_1, λ_2 runs from 0 to λ_1 and λ_3 runs from 0 to $\lambda_1 - \lambda_2$. The element of volume in these coordinates is

$$dV = \det\begin{pmatrix} \boldsymbol{w}_1^{\mathrm{T}} \\ \boldsymbol{w}_2^{\mathrm{T}} \\ \boldsymbol{w}_3^{\mathrm{T}} \end{pmatrix} d\lambda_1 \, d\lambda_2 \, d\lambda_3, \quad (33)$$

where the components of $\boldsymbol{w}_i^{\mathrm{T}}$ form the ith row in the matrix and the vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are labeled so that $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, and $\boldsymbol{w}_3$ form a right-handed triad. (See Gellert *et al* p. 466, Fulks pp. 454-6, Hildebrand pp. 306-9, and Arfken pp. 86-8.) The right-handedness insures that the determinant and, hence, the volume element is positive. Putting all of this together, the integral of a function $f(x_1, x_2, x_3)$ over the volume of a face with exterior vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ is given by

$$\int_{\frac{b}{a}}^{1} \int_{0}^{\lambda_1} \int_{0}^{\lambda_1 - \lambda_2} f\left(x_1(\lambda_1, \lambda_2, \lambda_3), x_2(\lambda_1, \lambda_2, \lambda_3), x_3(\lambda_1, \lambda_2, \lambda_3)\right) \det\begin{pmatrix} \boldsymbol{w}_1^{\mathrm{T}} \\ \boldsymbol{w}_2^{\mathrm{T}} \\ \boldsymbol{w}_3^{\mathrm{T}} \end{pmatrix} d\lambda_3 \, d\lambda_2 \, d\lambda_1 . \quad (34)$$

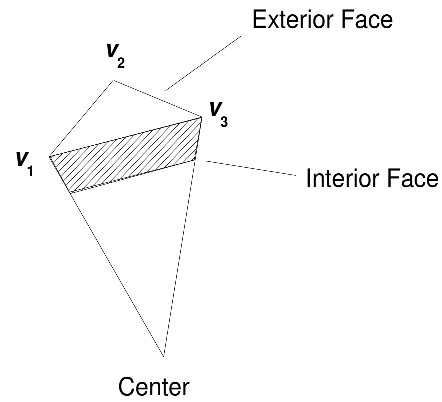

Figure 4: One face of an icosahedral shell

It is also useful to understand some properties of the exterior faces of icosahedra and how to integrate over them since this will be needed when calculating the effects of surface forces. It is convenient to use the parameters λ_1, λ_2, and λ_3 to describe the points on the exterior faces. Consider the exterior face with vertices $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, and $\boldsymbol{v_3}$. On the face, the parameter λ_1 has a fixed value of 1 and the parameters λ_2 and λ_3 range over the set of values that satisfies

$$0 \leqslant \lambda_2 + \lambda_3 \leqslant 1. \quad (35)$$

The unit normal to this face is

$$\hat{n} = \frac{\dfrac{\partial \boldsymbol{x}}{\partial \lambda_2} \times \dfrac{\partial \boldsymbol{x}}{\partial \lambda_3}}{\sqrt{EG - F^2}}, \quad (36)$$

where

$$\boldsymbol{w}_i = \sum_{j=1}^{3} w_{ij} \hat{e}_j \text{ for } i = 1,2,3, \quad \frac{\partial \boldsymbol{x}}{\partial \lambda_2} = \begin{pmatrix} w_{21} - w_{11} \\ w_{22} - w_{12} \\ w_{23} - w_{13} \end{pmatrix}, \quad \frac{\partial \boldsymbol{x}}{\partial \lambda_3} = \begin{pmatrix} w_{31} - w_{11} \\ w_{32} - w_{12} \\ w_{33} - w_{13} \end{pmatrix} \quad (37)$$

and

$$E = \frac{\partial \boldsymbol{x}}{\partial \lambda_2} \cdot \frac{\partial \boldsymbol{x}}{\partial \lambda_2}, \quad F = \frac{\partial \boldsymbol{x}}{\partial \lambda_2} \cdot \frac{\partial \boldsymbol{x}}{\partial \lambda_3}, \quad G = \frac{\partial \boldsymbol{x}}{\partial \lambda_3} \cdot \frac{\partial \boldsymbol{x}}{\partial \lambda_3}. \quad (38)$$

Due to the simple forms of the partial derivatives, the unit normal vector $\hat{n}$ is a constant vector and E, F, and G are constants over the entire face. This means that many of these quantities can be calculated once at the beginning of a numerical computation and stored for later use. The plane spanned by $\mathbf{v}_2$, and $\mathbf{v}_3$ and containing the point $\mathbf{v}_1$ is the tangent plane for every point on the face. And the element of area on the face in terms of the parameters λ_2 and λ_3 is

$$dS = \sqrt{EG - F^2}\, d\lambda_3\, d\lambda_2. \quad (39)$$

(See Gellert *et al* pp. 565-7, Fulks pp. 429-35, and Struik pp. 55-64.) So the integral of a function $f(x_1, x_2, x_3)$ over the exterior surface of a face with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is given by

$$\int_0^1 \int_0^{1-\lambda_2} f(x_1(1-\lambda_2-\lambda_3, \lambda_2, \lambda_3), x_2(1-\lambda_2-\lambda_3, \lambda_2, \lambda_3), x_3(1-\lambda_2-\lambda_3, \lambda_2, \lambda_3)) \sqrt{EG-F^2}\, d\lambda_3\, d\lambda_2. \quad (40)$$

Rather than describing the icosahedra by specifying the lengths of the outer and inner edges, it is probably more convenient for the current purposes to specify the length of the outer edges and the thickness of a face as measured, for example, in the center of the face. For an icosahedral shell of thickness T where the length of the outer edges is $2a$ and the length of the inner edges is $2b$,

$$b = a - \frac{T}{\sqrt{\frac{2}{3} + \tau}}, \quad (41)$$

where τ is the golden ratio = $(1+\sqrt{5})/2$. This formula makes it easy to convert back and forth between the two different ways of describing the icosahedra. The inner half edge length b must be greater than 0 and less than a to make physical sense or, equivalently, the thickness T must be less than the radius of the inscribed circle and greater than 0, i.e.

$$0 < T < a\sqrt{\frac{2}{3} + \tau}.$$

Physical Properties of Icosahedral Shells

Volume

The volume of an icosahedral shell that has outer edges of length $2a$ and inner edges of length $2b$ is

$$V = 10\left(1 + \frac{\sqrt{5}}{3}\right)\left(a^3 - b^3\right). \quad (42)$$

This is the volume of a solid icosahedron with edge length 2*a* minus the volume of a solid icosahedron with edge length 2*b*.

<u>Center of Mass</u>
By symmetry, the center of mass lies at the center of the body frame at (0,0,0), as the equations of motion derived above require.

<u>Principal Axes and Moment of Inertia Tensor</u>
The moment of inertia tensor of an icosahedral shell with outer edge length 2*a* and inner edge length 2*b* in the given body frame is

$$I = \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix}, \quad (43)$$

where

$$\Lambda = 4\left(\frac{7}{3}+\sqrt{5}\right)\rho\left(a^5-b^5\right). \quad (44)$$

Since the inertia tensor is diagonal, the given body coordinate axes are principal axes for the shell. (See Fowles pp. 230-2.) Furthermore, since all three diagonal components are equal, any Cartesian axes with the center of mass as their center will be principal axes. This is due to the symmetry of the shell and means that the resistance to rotation is the same around any axes that pass through the center of the shell.

The simple closed-form expression for the moment of inertia tensor for icosahedral shells is the primary motivation for using shells rather than a more general structure to represent icosahedra in the model. In general, evaluating the tensor requires the evaluation of a large number of volume integrals. However, since the inertia tensor is constant in the body coordinate frame, it only needs to be determined once for any given mass distribution. This could take place during initialization at the beginning of a simulation or exactly once outside of any simulation.

Despite the simple form of the inertia tensor for icosahedral shells, calculating it was rather complicated. It required the evaluation of 120 volume integrals – 6 for each face of the icosahedron. I used Maple for this purpose, checking the Maple calculations by performing the integrations over one of the faces by hand and integrating to find some known quantities that are calculated using similar integrals, namely the coordinates of the center of mass and the volume. Altogether, this required evaluating 200 integrals, which resulted in approximately 50 pages of Maple calculations. The Maple code is included in an appendix together with a summation of the output. (There is probably a simpler way to do this calculation. It is probably possible to evaluate the 6 integrations corresponding to one face and then use some physical principle to apply the results to the other faces.)

Equations of Motion for Icosahedral Materials

As described above, icosahedral materials are comprised of arrangements of icosahedra in space. Flexible bands connect the icosahedra and restrain their motions in all three spatial directions, tying the icosahedra together. Whatever the overall shape of a piece of material, the interconnecting bands generally join the icosahedra in regular patterns called “weaves”. The bulk properties of a material will depend upon the weave and the magnitudes and directions of application of any externally applied forces.

Each icosahedron in a material will move according to its equations of motion and a system of differential equations describing the rotation of its body reference frame relative to its center-of-mass frame

$$\begin{aligned} M\ddot{\boldsymbol{R}} &= \sum\nolimits_{icosahedron} \boldsymbol{F}, \\ \frac{d}{dt} I\vec{\omega} + \left(\vec{\omega}\times I\vec{\omega}\right) &= \sum\nolimits_{icosahedron} \boldsymbol{r}'\times\boldsymbol{F}, \\ \left(\frac{d\hat{e}_i}{dt}\right)_{cm} &= \vec{\omega}\times\hat{e}_i \ \text{ or } \ \frac{d\boldsymbol{q}}{dt} = \frac{1}{2}\boldsymbol{\omega}\boldsymbol{q}, \end{aligned} \qquad (45)$$

where $\boldsymbol{\omega}$ is the pure quaternion corresponding to the angular velocity vector $\vec{\omega}$, subject to the initial conditions

$$\begin{aligned} \boldsymbol{R}(0) &= \boldsymbol{R_0}, \\ \dot{\boldsymbol{R}}(0) &= \dot{\boldsymbol{R}}_{\boldsymbol{0}}, \\ \vec{\omega}(0) &= \vec{\omega}_0, \\ \hat{e}_i(0) = \sum_{j=1}^{3} \mu_{ij}(0)\hat{e}_j^0 \ &\text{ or } \ \boldsymbol{q}(0) = \boldsymbol{q}_0, \end{aligned} \qquad (46)$$

and under the influence of the externally applied and internally generated forces

$$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, t). \quad (47)$$

Suppose that there are *N* icosahedra in the material. For the k^{th} icosahedron, let $\boldsymbol{R}_k$ be the position of the center of mass, $\boldsymbol{r}'_k$ be the position from the center of mass to a point where a force is applied, $\vec{\omega}_k$ be the angular velocity, $\boldsymbol{\omega}_k$ be the pure quaternion associated with the angular velocity, $\hat{e}_{ik}$ be the i^{th} unit basis vector for the body frame, and $\boldsymbol{q}_k$ be the quaternion specifying the rotation of the body frame relative to the center of mass frame. Then the equations of motion for the elements of the material are

$$M\,\ddot{\boldsymbol{R}}_k \;=\; \sum\nolimits_{\text{icosahedron}\,k} \boldsymbol{F}_k\,,$$

$$\frac{d}{dt} I\,\vec{\omega}_k \;+\; \left(\vec{\omega}_k \times I\,\vec{\omega}_k\right) \;=\; \sum\nolimits_{\text{icosahedron}\,k} \boldsymbol{r}\,'_k \times \boldsymbol{F}_k\,,$$

$$\left(\frac{d\,\hat{e}_{ik}}{dt}\right)_{cmk} \;=\; \vec{\omega}_k \times \hat{e}_{ik} \;\text{ or }\; \frac{d\,\boldsymbol{q}_k}{dt} = \frac{1}{2}\boldsymbol{\omega}_k \boldsymbol{q}_k\,,$$

$$\text{for }\; k = 1, 2, \cdots, N\,,$$

with initial conditions

$$\boldsymbol{R}_k(0) = \boldsymbol{R}_{k0},$$

$$\dot{\boldsymbol{R}}_k(0) = \dot{\boldsymbol{R}}_{k0},$$

$$\vec{\omega}_k(0) = \vec{\omega}_{k0},$$

$$\hat{e}_i(0) = \sum_{j=1}^{3} \mu_{ij}(0)\,\hat{e}_j^{\,0} \;\text{ or }\; \boldsymbol{q}\,(0) = \boldsymbol{q}_{\boldsymbol{0}}$$

and where

$$\boldsymbol{F}_k = \boldsymbol{F}_k(\boldsymbol{r}_k, t)$$

represents the forces externally and internally applied to icosahedron *k*. The forces and moments in a static configuration will satisfy the equations

$$0 \;=\; \sum\nolimits_{\text{icosahedron}\,k} \boldsymbol{F}_k\,,$$

$$0 \;=\; \sum\nolimits_{\text{icosahedron}\,k} \boldsymbol{r}\,'_k \times \boldsymbol{F}_k, \quad (48)$$

$$\text{for}\, k = 1, 2, \cdots, N\,,$$

which simply state that the resultants of the forces and moments on the icosahedra are 0. In general, this is a coupled, nonlinear system of equations. Notice that the same equations pertain to any system of *N* rigid bodies. They describe the motions of an icosahedral material, in particular, when the pattern of connections and the connection force specify the internally generated forces, when *I* is the inertia tensor for an icosahedron, and when the resultants are calculated specifically for icosahedra.

Most typically, the initial configuration of an icosahedral material, which the initial conditions embody, is formed from the weave used to join the icosahedra with all of the interconnecting elements resting at their natural lengths. But this will not be true if the material is pre-stressed in some way.

In addition to the equations and initial conditions, there may be equality constraints that specify the positions, velocities, orientations, angular velocities, or combinations thereof for some of the icosahedra or possibly the positions of some interconnecting elements as functions of time. Some of these constraints may amount to boundary conditions imposed on the system. If constraints pertain to the positions of interconnecting elements, which are not explicitly represented in the model, then they operate by affecting the forces on the icosahedra attached to those connectors. When specifying such constraints, it is important to make sure that they do not contradict other conditions imposed on the system and that they remain consistent with the solution of the equations over time.

Aside from the equations and constraints described above, there are also constraints corresponding to the fact that solid icosahedral shells cannot interpenetrate. These are inequality constraints that must be taken into consideration in both static and dynamic calculations. As with the previous constraints, these must be consistent with the other conditions that pertain to the system and with the solutions of the equations. Unlike the previous constraints, they are complicated and difficult to formulate explicitly. In any simulation, they embody the task of detecting collisions. In the presence of these constraints, the determination of an equilibrium configuration of the system becomes a problem of minimizing the magnitudes of the force and moment resultants subject to all applicable constraints rather than a matter of solving the system of nonlinear equilibrium equations. A collision detection algorithm will define the feasible region corresponding to the collision constraints. If forces or moments persist after minimization, then the structure of the material must be capable of sustaining the calculated configuration for it to be physically realistic. In dynamic simulations, it is also necessary to specify how icosahedra behave when they collide. Specifying how icosahedra respond in a collision generally requires adding additional forces to any pairs of colliding icosahedra.

The issues related to the constraints described above also apply to any system of N rigid bodies. They specialize to the case of an icosahedral material when the particular bodies, forces, constraints, and responses are defined.

In the dynamic case, the overall system of equations that describes motions within the material is a system of $10N$ (or $15N$) ordinary differential equations together with the initial conditions, any constraints on individual icosahedra or connectors, and the collision constraints and responses. External forces may include the force of gravity, electromagnetic forces, applied compressions or extensions, or the forces of an impact. Internally generated forces result from the actions of interconnecting elements and possibly collisions between icosahedra. The dependence of forces on the position of icosahedra, in general, and the forces that interconnecting bands generate, in particular, which may, for example, take the form

$$F_{12}=k\left(\left|r_1-r_2\right|-D\right)\begin{cases}\dfrac{(r_2-r_1)}{\left|r_1-r_2\right|} & \text{on } I_1\\ \dfrac{(r_1-r_2)}{\left|r_1-r_2\right|} & \text{on } I_2\end{cases}, \quad (49)$$

couple the equations of motion of all of the icosahedra together, making it necessary to solve the entire 10 (or 15) N system of ODEs simultaneously. Similarly, the system that determines an equilibrium configuration consists of 10 (or 15) N nonlinear equations that must be solved simultaneously or 10 (or 15) N coupled, nonlinear quantities that must be minimized simultaneously under the given constraints.

Weaves

In what follows, I will assume that the connectors attach at the edges of the icosahedra. There are currently three different weaves under consideration: the orthogonal, the bias, and the radial. The radial weave is constructed using curved connectors that do not conform to my assumptions regarding how connection forces act. Therefore, I won't consider it further here. The orthogonal and bias weaves are actually identical except that they are rotations of each other relative to the primary reference frame or to the exterior boundaries of the material. See Figures 4 and 7, which show an icosahedron with all 12 possible connections oriented according to the orthogonal and bias weaves, respectively. (Early analysis and testing showed that orthogonal weave fabrics are unsuitable for building structures. However, ...) Although the orthogonal orientation has been found to not be of much practical use, it is easier to analyze than the bias orientation and most results concerning the orthogonal orientation may be translated into results concerning the bias orientation. Therefore, I will explore the orthogonal weave in detail below.

The Orthogonal Weave

In the orthogonal weave, all of the icosahedra are oriented identically with respect the primary reference frame, with a pair of opposite edges pointing directly up and directly down. For an individual icosahedron, the connectors attach at the 6 edges that intersect the circumscribing cube that contains the top and bottom edges. Each of these edges may have up to two connectors attached, for a total of up to 12 connections per icosahedron, depending on the location of the icosahedron in the material. If present, the two connectors run in roughly opposite directions from the edge where they attach. Connectors that attach to edges on opposite faces of the cube run along roughly parallel lines whereas connectors that attach to edges on adjacent faces of the cube run roughly perpendicular to each other. The two icosahedra that a connector joins lie on opposite sides of the connector. This creates spaces between the icosahedron and any other icosahedra in the directions of the connectors. Icosahedra at corners, edges, and sides of a material attach to the material via less than a full complement of 12 connectors whereas interior elements utilize all 12 potential connectors.

Figure 4: An icosahedron and the 12 icosahedra that connect to it in the orthogonal weave

Edge connections are conveniently thought of in terms of connections between the cubes that circumscribe the icosahedra and contain the 6 edges where the connectors attach. Let *L* be the length of the connectors, *w* be the width of the circumscribing cube, and θ be the angle between a connector and the circumscribing cube, where positive values of θ refer to situations where the connectors lie entirely outside of the circumscribing cube and negative values of θ refer to situations where the ends of the connectors lie within the circumscribing cubes. Then

$$w=2\tau a=(1+\sqrt{5})a \quad (50)$$

and, since the connectors meet the circumscribing cubes in the same manner in all directions, *L*, *w*, and $\sin\theta$ are related by the quadratic equation

$$2L^2\sin^2\theta+2wL\sin\theta-(L^2-w^2)=0. \quad (51)$$

Therefore,

$$\sin\theta = \frac{-w + \sqrt{2L^2 - w^2}}{2L}, \quad (52)$$

where

$$L \geq \frac{w}{\sqrt{2}} \approx .707106781\, w,$$

or

$$L = \left(\frac{\sin\theta + \cos\theta}{\cos 2\theta}\right) w \quad (53)$$

since $-\frac{\pi}{4} < \theta_{min} \leq \theta < \frac{\pi}{4}$, $L \geq 0$, $w \geq 0$, and $L = w$ when $\theta = 0$, where θ_{min} is the (negative) angle between the circumscribing cube and a face containing an edge on the cube, i.e.

$$\theta_{min} = -\arctan\left(\frac{\tau - 1}{\tau}\right) \approx -0.364863828 \text{ rad} = -20.90515745^\circ. \quad (54)$$

Figure 5 shows the connector length as a function of connection angle, where the blue short-dashed line on the left hand side denotes θ_{min} and the red long-dashed line on the right hand side is the asymptote at $\theta = 45^\circ$. The connector length L increases monotonically over the half-open interval $\theta_{min} \leq \theta < \frac{\pi}{4}$, taking on its minimum length

$$L_{min} \approx 0.774596669 w$$

at $\theta = \theta_{min}$. There is no maximum length and $L \to \infty$ as $\theta \to \frac{\pi}{4}$.

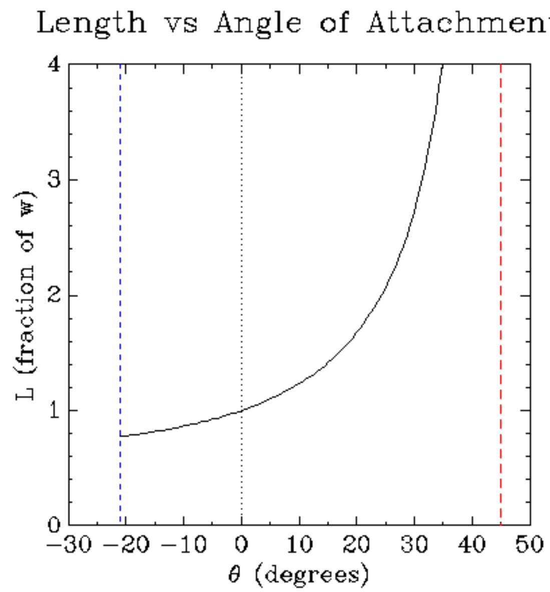

Figure 5: Connector length as a function of connection angle

In the body frame of any particular icosahedron in a fabric woven in the orthogonal weave, the centers of the icosahedra lie at

$$\boldsymbol{c}(i, j, k) = (w + L\sin\theta)\begin{pmatrix} 2i + (|j| + |k| \bmod 2) \bmod 2 \\ j \\ k \end{pmatrix}, \quad (55)$$

for integers *i*, *j*, and *k*, where the chosen icosahedron has indices $i = j = k = 0$ and mod 2 denotes arithmetic or reduction modulo 2. (See the corresponding Wolfram MathWorld entry or Behnke *et al*, pp. 174-6 for a definition of modular arithmetic.) Arithmetic modulo 2 can be implemented very efficiently on a typical computer using bit operations rather than division. This rule provides a specific numbering of the icosahedra in the fabric. The index *k* numbers the levels, which occupy planes perpendicular to the *z*-axis. Within a given level, the index *j* numbers the columns, which run parallel to the *x*-axis, and the index *i* numbers the icosahedra within each column. Since *i*, *j*, and *k* correspond to steps in the coordinate directions, I will call them "coordinate" indices. Notice, however, that the index *i* does not number the rows in the rule above, i.e. the centers of icosahedra in different columns labeled with the same value of *i* may not have the same *y*-component. The centers of icosahedra in neighboring columns with the same index i do not have the same x-component but those in the next nearest neighboring columns do.

Since the centers of the icosahedra are symmetrically positioned with respect to all 3 coordinate directions, rules that permute the coordinate indices i, j, and k and the corresponding components of the positions of the centers give equally legitimate rules for locating the centers of the icosahedra in the fabric in the body frame of the chosen icosahedron. For example, permuting i and j and the x and y-components of the above rule gives

$$\boldsymbol{c}(i,j,k) = (w + L\sin\theta)\begin{pmatrix} i \\ 2j + (|i| + |k| \bmod 2) \bmod 2 \\ k \end{pmatrix},$$

which also properly specifies the positions of the centers of the icosahedra. As before, the index k numbers the levels. But, in this case, the index i numbers the rows, which run parallel to the y-axis, and the index j numbers the icosahedra in the rows. Analogous to the situation for the previous rule, the index j does not number the columns and, in general, a triple of coordinate indices (i, j, k) will refer to different icosahedra under the two different rules.

A slab constructed out of the orthogonal weave is characterized by the number of rows N_x, the number of columns N_y, and the number of levels N_z of icosahedra it contains. Now, suppose that you want to construct a slab by filling a box with a fabric woven in the orthogonal weave, where the rows, columns, and levels of the fabric align with the sides of the box. First, define a triad of Cartesian coordinate axes aligned with the sides of the box and with the origin at a corner of the box, which will be taken to be the primary frame. To build a fabric to fill the box, you could start with either an icosahedron or a gap in the rear ($x = 0$), leftmost ($y = 0$), bottom ($z = 0$) corner of the box (when looking toward the origin from a point in the first octant). Furthermore, there are 2 distinct ways to orient the icosahedra. The basis vectors of the body frames of the icosahedra could point in the same directions as those of the primary frame or they could be rotated 90° about any coordinate direction compared to the basis vectors of the primary frame. These two orientations differ in the directions that the connectors extend from an edge relative to the primary frame. Any other orientation is equivalent to one of the two already described. If the basis vectors of the body frames of the icosahedra point in the directions of the basis vectors of the primary frame and the leftmost, rear, bottom corner contains an icosahedron, then the slab is said to be woven in the orthogonal weave, variant 0.

Alternatively, if the icosahedra are oriented in the same way and the leftmost, rear, bottom corner contains a gap rather than an icosahedron, then the slab is said to be woven in the orthogonal weave, variant 1. If the icosahedra in variant 0 or variant 1 are rotated 90°, then the slab is said to be woven in the orthogonal weave, variant 2 or 3, respectively. Correspondingly, for the icosahedra in variants 0 and 1, the quaternion that specifies the initial rotation of the body frames relative to the primary frame is the multiplicative identity

$$\boldsymbol{q}_0 = (1,0,0,0)$$

whereas the initial rotation quaternion for variants 2 and 3 is

$$\boldsymbol{q}_0 = \left(\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}\right),$$

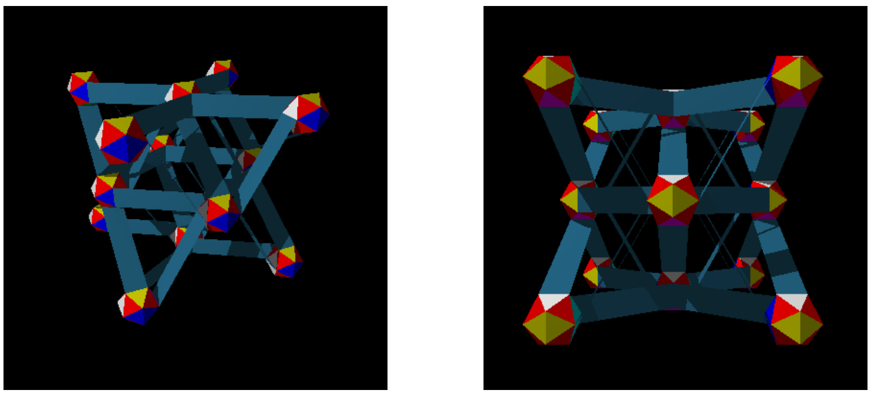

Illustration 6: The Orthogonal Weave, Variant 0. The left image shows a 3x3x3 slab from a point in front and to the left.

where I have chosen the z-axis as the axis of rotation.

If a slab is constructed in the **orthogonal weave, variant 0**, then the centers of the icosahedra lie at the points

$$\boldsymbol{c}(i,j,k) = (w + L\sin\theta)\begin{pmatrix} 2i + (j+k \bmod 2) \bmod 2 \\ j \\ k \end{pmatrix}, \quad (56)$$

for

$$i = 0,1,\cdots,N + [(j+1-k \bmod 2) \bmod 2](N_x \bmod 2) - 1;\quad j = 0,1,\cdots,N_y-1;\ k = 0,1,\cdots,N_z-1 \tag{57}$$

where

$$N = \frac{N_x - N_x \bmod 2}{2}, \tag{58}$$

which is $\frac{N_x}{2}$ without the fractional part. An alternative representation is

$$\boldsymbol{c}(i,j,k) = (w + L\sin\theta)\begin{pmatrix} i \\ 2j + (i+k \bmod 2) \bmod 2 \\ k \end{pmatrix},$$

for

$$i = 0,1,\cdots,N_x-1;$$
$$j = 0,1,\cdots,N + [(i+1-k \bmod 2) \bmod 2](N_y \bmod 2) - 1;$$
$$k = 0,1,\cdots,N_z-1$$

where

$$N = \frac{N_y - N_y \bmod 2}{2}.$$

Either of these rules may be used to determine the locations of the centers in a slab woven according to variant 0. Shifting the fabric in the top rule up one level replaces the icosahedron at the origin with a gap and shows that the centers in a slab laid out in the **orthogonal weave, variant 1** lie at

$$\boldsymbol{c}(i,j,k) = (w + L\sin\theta)\begin{pmatrix} 2i + [j + (k+1) \bmod 2] \bmod 2 \\ j \\ k \end{pmatrix}, \tag{59}$$

for

$$i = 0,1,\cdots,N + \{[j+1-(k+1) \bmod 2] \bmod 2\}(N_x \bmod 2) - 1;\quad j = 0,1,\cdots,N_y-1;\ k = 0,1,\cdots,N_z-1, \tag{60}$$

where

$$N = \frac{N_x - N_x \bmod 2}{2}.$$

Aside from specifying the locations of the centers, these formulas allow the icosahedra in a row, column, level, or slab to be counted. Tables 1, 2, 3, and 4 list the numbers of icosahedra in the various parts of a slab for all of the variants.

Variant	Both j and k odd or even	One of j or k odd and the other even
0 or 2	$\frac{1}{2}\left(N_x + N_x \bmod 2\right)$	$\frac{1}{2}\left(N_x - N_x \bmod 2\right)$
1 or 3	$\frac{1}{2}\left(N_x - N_x \bmod 2\right)$	$\frac{1}{2}\left(N_x + N_x \bmod 2\right)$

Table 1: Number of Icosahedra in a Column of a Slab Woven in the Orthogonal Weave (Here j is the column number, which might not be the same as the coordinate index j, and k is the level number.)

Variant	Both i and k odd or even	One of i or k odd and the other even
0 or 2	$\frac{1}{2}\left(N_y + N_y \bmod 2\right)$	$\frac{1}{2}\left(N_y - N_y \bmod 2\right)$
1 or 3	$\frac{1}{2}\left(N_y - N_y \bmod 2\right)$	$\frac{1}{2}\left(N_y + N_y \bmod 2\right)$

Table 2: Number of Icosahedra in a Row of a Slab Woven in the Orthogonal Weave (Here i is the row number, which might not be the same as the coordinate index i, and k is the level number.)

Variant	Number of Icosahedra in Level k
0 or 2	$\frac{1}{2}\left[N_x N_y + (-1)^{k+1}\left(N_x N_y\right) \bmod 2\right]$
1 or 3	$\frac{1}{2}\left[N_x N_y + (-1)^{k}\left(N_x N_y\right) \bmod 2\right]$

Table 3: Number of Icosahedra in a Level of a Slab Woven in the Orthogonal Weave

Variant 0 or 2	Variant 1 or 3
$\frac{1}{2}\left[N_x N_y N_z + \left(N_x N_y N_z\right) \bmod 2\right]$	$\frac{1}{2}\left[N_x N_y N_z - \left(N_x N_y N_z\right) \bmod 2\right]$

Table 4: Total Number of Icosahedra in a Slab Woven in the Orthogonal Weave

Since the orthogonal weave as described according to the rule above is invariant under mirror reflection about the *x*-*z* plane, any formula describing variant 0 is also true for variant 2 if the *x* and *y* components; *i* and *j*; and N_x and N_y are swapped. This may also be seen algebraically as follows. Rotating the fabric 90° about the z-axis in the top rule for variant 0 and reversing the direction of the index i demonstrates that the centers in a slab laid out in variant 2 lie at

$$\boldsymbol{c}(i,j,k) = (w + L\sin\theta)\begin{pmatrix} i \\ 2j + |-i + k \bmod 2| \bmod 2 \\ k \end{pmatrix},$$

for

$$i = 0, 1, \cdots, N_x - 1;$$
$$j = 0, 1, \cdots, N + (|1 - i - k \bmod 2| \bmod 2)(N_y \bmod 2) - 1;$$
$$k = 0, 1, \cdots, N_z - 1,$$

where, this time,

$$N = \frac{N_y - N_y \bmod 2}{2}.$$

It may easily be shown that

$$|-m + n| \bmod 2 = |m + n| \bmod 2$$

and

$$|m| \bmod 2 = |m + 2n| \bmod 2,$$

for any integers m and n. Applying these identities to the above formula shows that the centers of the icosahedra a slab woven in the **orthogonal weave, variant 2** lie at the points

$$\boldsymbol{c}(i,j,k) = (w + L\sin\theta)\begin{pmatrix} i \\ 2j + (i + k \bmod 2) \bmod 2 \\ k \end{pmatrix}, \quad (61)$$

for

$$i = 0, 1, \cdots, N_x - 1;$$
$$j = 0, 1, \cdots, N + [(i + 1 - k \bmod 2) \bmod 2](N_y \bmod 2) - 1; \quad (62)$$
$$k = 0, 1, \cdots, N_z - 1.$$

Notice that this is the same formula as the bottom formula describing the locations of the centers for variant 0. In other words, the centers lie in the same locations for variants 0 and 2 of the orthogonal weave. This also means that slabs with the same dimensions in variants 0 and 2 have the same numbers of icosahedra in their corresponding rows, columns, and levels and the same total numbers of icosahedra. See Tables 1, 2, 3, and 4.

Similarly, the centers of the icosahedra are in the same positions for variants 1 and 3.

Shifting the rule describing the locations of the centers in variant 2 yields a rule describing the locations of the centers in a slab woven using variant 3. This shows that the centers are located at

$$\boldsymbol{c}(i,j,k) = (w + L\sin\theta)\begin{pmatrix} i \\ 2j + [i + (k+1)\bmod 2]\bmod 2 \\ k \end{pmatrix}, \quad (63)$$

for

$$\begin{gathered} i = 0,1,\cdots,N_x - 1; \\ j = 0,1,\cdots,N + \{[i + 1 - (k+1)\bmod 2]\bmod 2\}(N_y \bmod 2) - 1; \quad (64) \\ k = 0,1,\cdots,N_z - 1 \end{gathered}$$

if the slab is laid out in the **orthogonal weave, variant 3**, where, again,

$$N = \frac{N_y - N_y \bmod 2}{2}. \quad (65)$$

This is an alternative form for the positions of the centers of the icosahedra for variant 1. As with variants 0 and 2, in slabs constructed in variants 1 and 3 with the same dimensions, the locations of the centers; the numbers of icosahedra in corresponding rows, columns, and levels; and the total numbers of icosahedra are the same. See Tables 1, 2, 3, and 4.

Rather than using the 3 indices *i*, *j*, and *k* to label the icosahedra in a slab, it is also possible to label the icosahedra using a single index α. One common way of doing this is through a lexicographic ordering. For example, if the icosahedra in a slab are laid out in the **orthogonal weave, variant 0**, and the top rule is used to determine the locations of the centers, then the index α takes its lowest value at the rear, leftmost, bottom corner and counts the icosahedra along the first (i.e. leftmost) column on the bottom, then the second column on the bottom, and so forth until the entire level has been counted. The icosahedra on each successive level are counted in the same manner until all of the icosahedra in the slab have been counted. In this case, the lexicographic index α is related to the coordinate indices *i*, *j*, and *k* via the formula

$$\alpha = \frac{N_x N_y k + (N_x N_y k)\bmod 2}{2} + \frac{N_x j + (-1)^k (N_x j)\bmod 2}{2} + i,$$

where α starts at 0, just like *i*, *j*, and *k*. Similarly, under the rule specifying the locations of the centers in a slab woven in the **orthogonal weave, variant 1**, the two types of indices satisfy

$$\alpha = \frac{N_x N_y k - (N_x N_y k) \bmod 2}{2} + \frac{N_x j + (-1)^{k+1} (N_x j) \bmod 2}{2} + i.$$

For the rules written above that give the locations of the centers for the remaining two variants, it is more convenient to define the lexicographic ordering slightly differently. In these cases, the lexicographic index α is incremented along the rows of each level rather than the columns. So, for the **orthogonal weave, variant 2**,

$$\alpha = \frac{N_x N_y k + (N_x N_y k) \bmod 2}{2} + \frac{i N_y + (-1)^{k} (i N_y) \bmod 2}{2} + j$$

whereas, for the **orthogonal weave, variant 3**,

$$\alpha = \frac{N_x N_y k - (N_x N_y k) \bmod 2}{2} + \frac{i N_y + (-1)^{k+1} (i N_y) \bmod 2}{2} + j.$$

Notice that these formulas also pertain to alternative rules for specifying the locations of the centers for variants 0 and 1, respectively. Correspondingly, the lexicographic orderings given for variants 0 and 1 could be used for variants 2 and 3, respectively, if the rules for locating the centers in variants 0 and 1 are used to describe variants 2 and 3 instead.

For the purposes of describing connections between icosahedra, it is useful to name the edges where the connectors attach. In the orthogonal and bias weaves, the axes of the body frame of an icosahedron pass through the edges where the connections attach. I will call the edges that the positive and negative *z*-axes pass through the "up" and "down" edges, respectively. The edges that the positive and negative *x*-axes pass through will be called the "north" and "south" edges, respectively. And, the edges that the positive and negative *y*-axes pass through will be called the "west" and "east" edges, respectively. As I mentioned before, the orthogonal and bias weaves are fundamentally identical and the connections within them are identical when seen from the perspective of the body frames of the connected icosahedra. Up edges always connect to down edges and vice versa; north edges always connect to south edges and vice versa; and west edges always connect to east edges and vice versa. (A diagram would be nice.) The locations of the centers of the icosahedra connected to a given icosahedron relative to the center of the given icosahedron are listed in Table 5.

Local Edge	Remote Edge	Location of Center of Remote Icosahedron
up	down	$+\Delta z, \pm\Delta x$
down	up	$-\Delta z, \pm\Delta x$
north	south	$+\Delta x, \pm\Delta y$
south	north	$-\Delta x, \pm\Delta y$
west	east	$+\Delta y, \pm\Delta z$
east	west	$-\Delta y, \pm\Delta z$

Table 5: Locations of Centers of ("Remote") Icosahedra Connected to a Given ("Local") Icosahedron Relative to its Center. Here $\Delta x = \Delta y = \Delta z = w + L\sin\theta$.

Local Edge	Remote Edge	$\Delta \boldsymbol{c}$	both *k* and *j* odd or even	one of *k* or *j* odd and the other even
up	down	$(W, 0, W)$	$(i, j, k+1)$	$(i+1, j, k+1)$
up	down	$(-W, 0, W)$	$(i-1, j, k+1)$	$(i, j, k+1)$
west	east	$(0, W, W)$	$(i, j+1, k+1)$	$(i, j+1, k+1)$
east	west	$(0, -W, W)$	$(i, j-1, k+1)$	$(i, j-1, k+1)$
down	up	$(W, 0, -W)$	$(i, j, k-1)$	$(i+1, j, k-1)$
down	up	$(-W, 0, -W)$	$(i-1, j, k-1)$	$(i, j, k-1)$
west	east	$(0, W, -W)$	$(i, j+1, k-1)$	$(i, j+1, k-1)$
east	west	$(0, -W, -W)$	$(i, j-1, k-1)$	$(i, j-1, k-1)$
north	south	$(W, W, 0)$	$(i, j+1, k)$	$(i+1, j+1, k)$
south	north	$(-W, W, 0)$	$(i-1, j+1, k)$	$(i, j+1, k)$
north	south	$(W, -W, 0)$	$(i, j-1, k)$	$(i+1, j-1, k)$
south	north	$(-W, -W, 0)$	$(i-1, j-1, k)$	$(i, j-1, k)$

Table 6: Possible connections in the orthogonal weave, variant 0. This table lists the coordinate indices of "remote" icosahedra that are connected to the "local" icosahedron with coordinate indices (i, j, k), provided that they are present in the fabric. Here k is the level containing the local icosahedron, j is the column that it's in, and $\Delta \boldsymbol{c}$ *is the relative position of the center of the remote icosahedron, where* $W \equiv w + L\sin\theta$. *This assumes that the coordinate indices are defined according to the "top" rule specifying the locations of the centers of the icosahedra in the slab.*

Local Edge	Remote Edge	$\Delta \boldsymbol{c}$	both *k* and *j* odd or even	one of *k* or *j* odd and the other even
up	down	$(W, 0, W)$	$(i+1, j, k+1)$	$(i, j, k+1)$
up	down	$(-W, 0, W)$	$(i, j, k+1)$	$(i-1, j, k+1)$
west	east	$(0, W, W)$	$(i, j+1, k+1)$	$(i, j+1, k+1)$
east	west	$(0, -W, W)$	$(i, j-1, k+1)$	$(i, j-1, k+1)$
down	up	$(W, 0, -W)$	$(i+1, j, k-1)$	$(i, j, k-1)$
down	up	$(-W, 0, -W)$	$(i, j, k-1)$	$(i-1, j, k-1)$
west	east	$(0, W, -W)$	$(i, j+1, k-1)$	$(i, j+1, k-1)$
east	west	$(0, -W, -W)$	$(i, j-1, k-1)$	$(i, j-1, k-1)$
north	south	$(W, W, 0)$	$(i+1, j+1, k)$	$(i, j+1, k)$
south	north	$(-W, W, 0)$	$(i, j+1, k)$	$(i-1, j+1, k)$
north	south	$(W, -W, 0)$	$(i+1, j-1, k)$	$(i, j-1, k)$
south	north	$(-W, -W, 0)$	$(i, j-1, k)$	$(i-1, j-1, k)$

Table 7: Possible connections in the orthogonal weave, variant 1. This table lists the coordinate indices of "remote" icosahedra that are connected to the "local" icosahedron with coordinate indices (i, j, k), provided that they are present in the fabric. Here k is the level containing the local icosahedron, j is the column that it's in, and $\Delta \boldsymbol{c}$ *is the relative position of the center of the remote icosahedron, where* $W \equiv w + L \sin\theta$ *. This assumes that the coordinate indices are defined according to the rule specifying the locations of the centers of the icosahedra in the slab.*

Local Edge	Remote Edge	$\Delta \mathbf{c}$	both *k* and *i* odd or even	one of *k* or *i* odd and the other even
up	down	$(0, W, W)$	$(i, j, k+1)$	$(i, j+1, k+1)$
up	down	$(0, -W, W)$	$(i, j-1, k+1)$	$(i, j, k+1)$
east	west	$(W, 0, W)$	$(i+1, j, k+1)$	$(i+1, j, k+1)$
west	east	$(-W, 0, W)$	$(i-1, j, k+1)$	$(i-1, j, k+1)$
down	up	$(0, W, -W)$	$(i, j, k-1)$	$(i, j+1, k-1)$
down	up	$(0, -W, -W)$	$(i, j-1, k-1)$	$(i, j, k-1)$
east	west	$(W, 0, -W)$	$(i+1, j, k-1)$	$(i+1, j, k-1)$
west	east	$(-W, 0, -W)$	$(i-1, j, k-1)$	$(i-1, j, k-1)$
north	south	$(W, W, 0)$	$(i+1, j, k)$	$(i+1, j+1, k)$
south	north	$(W, -W, 0)$	$(i+1, j-1, k)$	$(i+1, j, k)$
north	south	$(-W, W, 0)$	$(i-1, j, k)$	$(i-1, j+1, k)$
south	north	$(-W, -W, 0)$	$(i-1, j-1, k)$	$(i-1, j, k)$

Table 8: Possible connections in the orthogonal weave, variant 2. This table lists the coordinate indices of "remote" icosahedra that are connected to the "local" icosahedron with coordinate indices (i, j, k), provided that they are present in the fabric. Here k is the level containing the local icosahedron, i is the row that it's in, and $\Delta \mathbf{c}$ *is the relative position of the center of the remote icosahedron, where* $W \equiv w + L \sin\theta$ *. This assumes that the coordinate indices are defined according to the rule specifying the locations of the centers of the icosahedra in the slab.*

Local Edge	Remote Edge	$\Delta \boldsymbol{c}$	both *k* and *i* odd or even	one of *k* or *i* odd and the other even
up	down	$(0, W, W)$	$(i, j+1, k+1)$	$(i, j, k+1)$
up	down	$(0, -W, W)$	$(i, j, k+1)$	$(i, j-1, k+1)$
east	west	$(W, 0, W)$	$(i+1, j, k+1)$	$(i+1, j, k+1)$
west	east	$(-W, 0, W)$	$(i-1, j, k+1)$	$(i-1, j, k+1)$
down	up	$(0, W, -W)$	$(i, j+1, k-1)$	$(i, j, k-1)$
down	up	$(0, -W, -W)$	$(i, j, k-1)$	$(i, j-1, k-1)$
east	west	$(W, 0, -W)$	$(i+1, j, k-1)$	$(i+1, j, k-1)$
west	east	$(-W, 0, -W)$	$(i-1, j, k-1)$	$(i-1, j, k-1)$
north	south	$(W, W, 0)$	$(i+1, j+1, k)$	$(i+1, j, k)$
south	north	$(W, -W, 0)$	$(i+1, j, k)$	$(i+1, j-1, k)$
north	south	$(-W, W, 0)$	$(i-1, j+1, k)$	$(i-1, j, k)$
south	north	$(-W, -W, 0)$	$(i-1, j, k)$	$(i-1, j-1, k)$

Table 9: Possible connections in the orthogonal weave, variant 3. This table lists the coordinate indices of "remote" icosahedra that are connected to the "local" icosahedron with coordinate indices (i, j, k), provided that they are present in the fabric. Here k is the level containing the local icosahedron, i is the row that it's in, and $\Delta \boldsymbol{c}$ *is the relative position of the center of the remote icosahedron, where* $W \equiv w + L \sin\theta$ *. This assumes that the coordinate indices are defined according to the rule specifying the locations of the centers of the icosahedra in the slab.*

Determining how many connections there are in a slab woven in a particular variant of the orthogonal weave requires determining which possible connected neighbors of each icosahedron in the slab are actually present in the slab.

Variant	Number of Connections in a Slab
0 or 1	$(N_z-1)\left\{\frac{1}{2}\left(N_y-N_y \bmod 2\right)\left[8+12(N-1)+6(N_x \bmod 2)\right] + 4(N_y \bmod 2)-3+\left[6(N_y \bmod 2)-4\right](N-1) + \left[3(N_y \bmod 2)-2\right](N_x \bmod 2)\right\} + (N_y-1)\left[1+2(N-1)+N_x \bmod 2\right]$ where $N\equiv\frac{1}{2}\left(N_x-N_x \bmod 2\right)$
2 or 3	$(N_z-1)\left\{\frac{1}{2}\left(N_x-N_x \bmod 2\right)\left[8+12(N-1)+6(N_y \bmod 2)\right] + 4(N_x \bmod 2)-3+\left[6(N_x \bmod 2)-4\right](N-1) + \left[3(N_x \bmod 2)-2\right](N_y \bmod 2)\right\} + (N_x-1)\left[1+2(N-1)+N_y \bmod 2\right]$ where $N\equiv\frac{1}{2}\left(N_y-N_y \bmod 2\right)$

Table 10: Numbers of Connections in Slabs Woven from the Orthogonal Weave

The depth of a slab woven in any variant of the orthogonal weave in the *x*, *y*, and *z* directions of the body frame of any icosahedron in the slab, respectively, are

$$D_x=w+\left(N_x-1\right)(w+L\sin\theta),$$
$$D_y=w+\left(N_y-1\right)(w+L\sin\theta),$$
$$D_z=w+\left(N_z-1\right)(w+L\sin\theta),$$

where $w=2\tau a$ is the width of the circumscribing cube. Correspondingly, the numbers of icosahedra in the coordinate directions, N_x, N_y, and N_z, that best fit a cuboid with edge lengths W_x, W_y, and W_z are

$$N_x=\left[1+\frac{W_x-w}{w+L\sin\theta}\right],\quad N_y=\left[1+\frac{W_y-w}{w+L\sin\theta}\right],\quad N_z=\left[1+\frac{W_z-w}{w+L\sin\theta}\right],$$

where the square brackets mean “the largest integer smaller than the quantity they contain.”

The Bias Weave

To form the bias weave, the orthogonal weave is rotated so that opposite faces of the icosahedra point directly up and directly down. This rotates and tilts the circumscribing cubes and slants the restoring forces that the connecting elements exert. It also reorients the spaces in the material so that they are no longer directly above, below, or to the sides of the icosahedra.

Refer to the “bias orientation” rather than the “bias weave”.

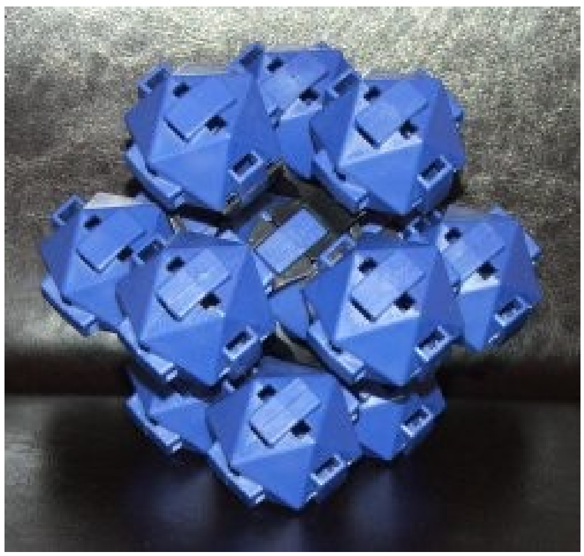

Figure 7: An icosahedron and the 12 icosahedra that connect to it in the bias weave

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Appendix

This appendix contains a listing of the Maple code used to calculate the volume, center of mass, and moment of inertia tensor for an icosahedral shell together with some of its output.

---------------------------- Maple output ----------------------------

```
>  read 'C:\\My Documents\\icosahedron\\icosahedron.txt';
```

$$V\_total := 10\,a^{-3} + \frac{10}{3}\,a^{-3}\sqrt{5} - 10\,b^3 - \frac{10}{3}\,b^3\sqrt{5}$$

$$xcm\_total := 0$$

$$ycm\_total := 0$$

$$zcm\_total := 0$$

$$Ixx\_total := -\frac{16}{3}\,\frac{\rho\,(-7\,a^{-5} + 7\,b^5 - 3\sqrt{5}\,a^{-5} + 3\sqrt{5}\,b^5)}{(-1+\sqrt{5})\,(1+\sqrt{5})}$$

$$Iyy\_total := -\frac{16}{3}\,\frac{\rho\,(-7\,a^{-5} + 7\,b^5 - 3\sqrt{5}\,a^{-5} + 3\sqrt{5}\,b^5)}{(-1+\sqrt{5})\,(1+\sqrt{5})}$$

$$Izz\_total := -\frac{16}{3}\,\frac{\rho\,(-7\,a^{-5} + 7\,b^5 - 3\sqrt{5}\,a^{-5} + 3\sqrt{5}\,b^5)}{(-1+\sqrt{5})\,(1+\sqrt{5})}$$

$$Ixy\_total := 0$$

$$Ixz\_total := 0$$

$$Iyz\_total := 0$$

$$Inertia := \begin{bmatrix} \%1 & 0 & 0 \\ 0 & \%1 & 0 \\ 0 & 0 & \%1 \end{bmatrix}$$

$$\%1 := -\frac{16}{3}\,\frac{\rho\,(-7\,a^{-5} + 7\,b^5 - 3\sqrt{5}\,a^{-5} + 3\sqrt{5}\,b^5)}{(-1+\sqrt{5})\,(1+\sqrt{5})}$$

$$Esystem := [\frac{16}{3}\,\frac{\rho\,(-a^{-5} + b^5)}{-7 + 3\sqrt{5}},\ 3,\ \{[0, 0, 1], [0, 1, 0], [1, 0, 0]\}]$$

---------------------------- end output ----------------------------

```
-----------------------------begin Maple listing -----------------------------
# This script calculates properties of a concentric icosahedral shell of
# uniform density rho, where the length of an edge of the outer icosahedral
# surface is a and the length of an edge of the inner icosahedral surface
# is b.  The coordinate system used has its origin at the center of the
# shell with the axes lying in the centers of the 3 perpendicular golden
# rectangles formed by 6 edges of the icosahedron.  See the Wikipedia
# entry for "icosahedron" (http://en.wikipedia.org/wiki/Icosahedron) for
# a picture of these rectangles.  The parameter tau is the golden ratio
# = (1 + sqrt(5))/2.  (See http://en.wikipedia.org/wiki/Golden_ratio
# for more information.)
#
# The script calculates the volume of the shell V, which is the volume
# of the outer icosahedron minus the volume of the inner icosahedron,
# and the center of mass (xcm, ycm, zcm), which lies at the center of
# the shell, as checks to indicate whether the integrations are being
# performed correctly.  But the purpose of the script is to calculate
# the components of the moment of inertia tensor I and find the principal
# axes for the shell.
#
# Mark A. Martin
# 6/2/2006

with(linalg):

assume(a>0):

# List the lists of vertices specifying the faces.

Faces := [
      [[0, -a, tau*a],  [0,a,tau*a],     [-tau*a,0,a]],    #  1
      [[0, -a, tau*a],  [tau*a,0,a],     [0,a,tau*a]],     #  2
      [[0, -a, tau*a],  [-tau*a,0,a],    [-a,-tau*a,0]],   #  3
      [[0, -a, tau*a],  [-a,-tau*a,0],   [a,-tau*a,0]],    #  4
      [[0, -a, tau*a],  [a,-tau*a,0],    [tau*a,0,a]],     #  5
      [[0, a, tau*a],   [-a,tau*a,0],    [-tau*a,0,a]],    #  6
      [[0, a, tau*a],   [a,tau*a,0],     [-a,tau*a,0]],    #  7
```

```
    [[0, a, tau*a],   [tau*a,0,a],     [a,tau*a,0]],     #  8
    [[-tau*a, 0, a],  [-tau*a, 0, -a], [-a, -tau*a, 0]], #  9
    [[-tau*a, 0, a],  [-a, tau*a, 0],  [-tau*a, 0, -a]], # 10
    [[tau*a, 0, a],   [a, -tau*a, 0],  [tau*a, 0, -a]],  # 11
    [[tau*a, 0, a],   [tau*a, 0, -a],  [a, tau*a, 0]],   # 12
    [[0, -a, -tau*a], [-a,-tau*a,0],   [-tau*a,0,-a]],   # 13
    [[0, -a, -tau*a], [a,-tau*a,0],    [-a,-tau*a,0]],   # 14
    [[0, -a, -tau*a], [tau*a,0,-a],    [a,-tau*a,0]],    # 15
    [[0, a, -tau*a],  [-tau*a,0,-a],   [-a,tau*a,0]],    # 16
    [[0, a, -tau*a],  [-a,tau*a,0],    [a,tau*a,0]],     # 17
    [[0, a, -tau*a],  [a,tau*a,0],     [tau*a,0,-a]],    # 18
    [[0, -a, -tau*a], [-tau*a,0,-a],   [0,a,-tau*a]],    # 19
    [[0, a, -tau*a],  [tau*a,0,-a],    [0,-a,-tau*a]]    # 20
   ]:

# Form a set of the lists of vertices specifying the faces to check that
# they are all unique.

Set_of_Faces := {
    {[0, -a, tau*a],  [0,a,tau*a],     [-tau*a,0,a]},    #  1
    {[0, -a, tau*a],  [0,a,tau*a],     [tau*a,0,a]},     #  2
    {[0, -a, tau*a],  [-tau*a,0,a],    [-a,-tau*a,0]},   #  3
    {[0, -a, tau*a],  [-a,-tau*a,0],   [a,-tau*a,0]},    #  4
    {[0, -a, tau*a],  [a,-tau*a,0],    [tau*a,0,a]},     #  5
    {[0, a, tau*a],   [-tau*a,0,a],    [-a,tau*a,0]},    #  6
    {[0, a, tau*a],   [-a,tau*a,0],    [a,tau*a,0]},     #  7
    {[0, a, tau*a],   [a,tau*a,0],     [tau*a,0,a]},     #  8
    {[-tau*a, 0, a],  [-a, -tau*a, 0], [-tau*a, 0, -a]}, #  9
    {[-tau*a, 0, a],  [-a, tau*a, 0],  [-tau*a, 0, -a]}, # 10
    {[tau*a, 0, a],   [a, -tau*a, 0],  [tau*a, 0, -a]},  # 11
    {[tau*a, 0, a],   [a, tau*a, 0],   [tau*a, 0, -a]},  # 12
    {[0, -a, -tau*a], [-tau*a,0,-a],   [-a,-tau*a,0]},   # 13
    {[0, -a, -tau*a], [-a,-tau*a,0],   [a,-tau*a,0]},    # 14
    {[0, -a, -tau*a], [a,-tau*a,0],    [tau*a,0,-a]},    # 15
    {[0, a, -tau*a],  [-tau*a,0,-a],   [-a,tau*a,0]},    # 16
    {[0, a, -tau*a],  [-a,tau*a,0],    [a,tau*a,0]},     # 17
    {[0, a, -tau*a],  [a,tau*a,0],     [tau*a,0,-a]},    # 18
```

```
        {[0, -a, -tau*a], [0,a,-tau*a],    [-tau*a,0,-a]},   # 19
        {[0, a, -tau*a],  [0,-a,-tau*a],   [tau*a,0,-a]}     # 20
        }:

NF := nops(Set_of_Faces);

if NF <> 20 then
   # Print an error message and terminate execution.
   ERROR(`There must be 20 unique faces!`);
fi;

Lengths := matrix(NF,3):
scale_factor := vector(NF):
V       := vector(NF):
xcm     := vector(NF):
ycm     := vector(NF):
zcm     := vector(NF):
Ixx     := vector(NF):
Iyy     := vector(NF):
Izz     := vector(NF):
Ixy     := vector(NF):
Ixz     := vector(NF):
Iyz     := vector(NF):

V_total   := 0:
xcm_total := 0:
ycm_total := 0:
zcm_total := 0:
Ixx_total := 0:
Iyy_total := 0:
Izz_total := 0:
Ixy_total := 0:
Ixz_total := 0:
Iyz_total := 0:

for i from 1 to NF do
```

```
P := Faces[i];

# Construct the basis vectors for integration.

v1 := vector(3, P[1]);
v2 := vector(3, P[2] - P[1]);
v3 := vector(3, P[3] - P[1]);

# Check the lengths of the edges of the outer face to make sure
# that they are all 2a.

tau := (1 + sqrt(5))/2:

Lengths[i,1] := simplify(sqrt(dotprod(v2,v2)));
Lengths[i,2] := simplify(sqrt(dotprod(v3,v3)));
Lengths[i,3] := simplify(sqrt(dotprod(v3-v2,v3-v2)));

# Construct the transformation matrix.
# The vectors v1, v2, and v3 must form a right-handed triad for
# the scale factor to have the correct sign.  I chose the ordering
# of the points in the faces of the icosahedron (as listed in
# Faces and Sets_of_Faces) by hand to make sure that this was true.

tau := 'tau':

scale_factor[i] := simplify(dotprod(crossprod(v1,v2),v3));

A := augment(v1,augment(v2,v3)):

# Express the position vector as a function of the face coordinates
# and retrieve its cartesian coordinates.

r := multiply(A, [lambda_1, lambda_2, lambda_3]):

x := r[1];
y := r[2];
z := r[3];
```

```
# Calculate volume (as a check).

V[i] := simplify(int(int(int(scale_factor[i],
                     lambda_3=0..(lambda_1 - lambda_2)),
                     lambda_2=0..lambda_1),
                     lambda_1=(b/a)..1));

V_total := V_total + V[i]:

# Calculate the coordinates of the center of mass (also as a check).
# See Fowles p 187.

xcm[i] := simplify(int(int(int(x*scale_factor[i],
                     lambda_3=0..(lambda_1 - lambda_2)),
                     lambda_2=0..lambda_1),
                     lambda_1=(b/a)..1));

xcm_total := xcm_total + xcm[i]:

ycm[i] := simplify(int(int(int(y*scale_factor[i],
                     lambda_3=0..(lambda_1 - lambda_2)),
                     lambda_2=0..lambda_1),
                     lambda_1=(b/a)..1));

ycm_total := ycm_total + ycm[i]:

zcm[i] := simplify(int(int(int(z*scale_factor[i],
                     lambda_3=0..(lambda_1 - lambda_2)),
                     lambda_2=0..lambda_1),
                     lambda_1=(b/a)..1));

zcm_total := zcm_total + zcm[i]:

# Calculate the components of the moment of inertia tensor.
# See Fowles p 219.
```

```
Ixx[i] := int(int(int((y*y + z*z)*rho*scale_factor[i],
             lambda_3=0..(lambda_1 - lambda_2)),
             lambda_2=0..lambda_1),
             lambda_1=(b/a)..1):

Iyy[i] := int(int(int((x*x + z*z)*rho*scale_factor[i],
             lambda_3=0..(lambda_1 - lambda_2)),
             lambda_2=0..lambda_1),
             lambda_1=(b/a)..1):

Izz[i] := int(int(int((x*x + y*y)*rho*scale_factor[i],
             lambda_3=0..(lambda_1 - lambda_2)),
             lambda_2=0..lambda_1),
             lambda_1=(b/a)..1):

Ixy[i] := int(int(int(-x*y*rho*scale_factor[i],
             lambda_3=0..(lambda_1 - lambda_2)),
             lambda_2=0..lambda_1),
             lambda_1=(b/a)..1):

Ixz[i] := int(int(int(-x*z*rho*scale_factor[i],
             lambda_3=0..(lambda_1 - lambda_2)),
             lambda_2=0..lambda_1),
             lambda_1=(b/a)..1):

Iyz[i] := int(int(int(-y*z*rho*scale_factor[i],
             lambda_3=0..(lambda_1 - lambda_2)),
             lambda_2=0..lambda_1),
             lambda_1=(b/a)..1):

tau := (1 + sqrt(5))/2:

Ixx[i] := simplify(eval(Ixx[i]));
Iyy[i] := simplify(eval(Iyy[i]));
Izz[i] := simplify(eval(Izz[i]));
Ixy[i] := simplify(eval(Ixy[i]));
Ixz[i] := simplify(eval(Ixz[i]));
```

```
    Iyz[i] := simplify(eval(Iyz[i]));

    Ixx_total := Ixx_total + Ixx[i]:
    Iyy_total := Iyy_total + Iyy[i]:
    Izz_total := Izz_total + Izz[i]:
    Ixy_total := Ixy_total + Ixy[i]:
    Ixz_total := Ixz_total + Ixz[i]:
    Iyz_total := Iyz_total + Iyz[i]:

    tau := 'tau':

od;

tau := (1 + sqrt(5))/2:

V_total   := simplify(V_total);
xcm_total := simplify(xcm_total);
ycm_total := simplify(ycm_total);
zcm_total := simplify(zcm_total);
Ixx_total := simplify(Ixx_total);
Iyy_total := simplify(Iyy_total);
Izz_total := simplify(Izz_total);
Ixy_total := simplify(Ixy_total);
Ixz_total := simplify(Ixz_total);
Iyz_total := simplify(Iyz_total);

Inertia := matrix(3,3,[[Ixx_total, Ixy_total, Ixz_total],
                       [Ixy_total, Iyy_total, Iyz_total],
                       [Ixz_total, Iyz_total, Izz_total]]);

Esystem := eigenvects(Inertia);
```
----------------------------- end Maple listing -----------------------------